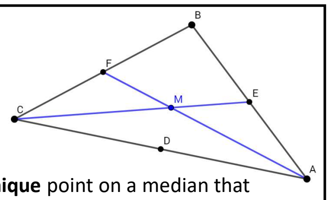
Goal

- Proof Summary
  - Prove that the centroid exists
  - Prove that the circumcenter exists
  - Prove that the orthocenter lies on the line formed by the centroid and circumcenter
- This completes the proof of the Euler Line

Beginning Proof

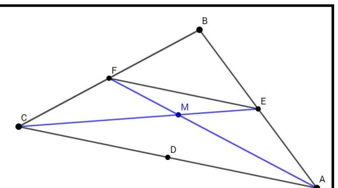
- We start our proof by drawing an arbitrary triangle
  - Let our triangle be called ΔABC
  - Let the midpoints of the triangle be E, F, D
- This is important because:
  - We want to show that each of the centers and the Euler Line exists **for all** possible triangles

Centroid



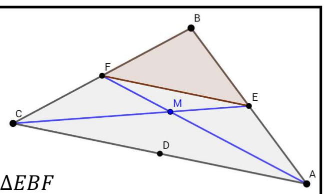
- There is only one **unique** point on a median that splits the median into a 1:2 ratio (Midpoint-Vertex)
- To show the centroid exists, we will show that:
  - Any two medians will **always** intersect at a point that splits the other median in a 1:2 ratio (Midpoint-Vertex)
- Hence, if the above statement is true, then the third median **must** intersect the other medians at the exact same point

Centroid



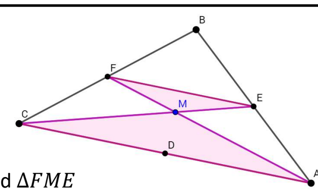
- First, we arbitrarily draw the medians CE and AF
  - We could start with any 2 of the 3 medians
- Let M be the intersection of CE and AF
- Next, we draw the line FE
- We will show that $CM = 2ME$ and $AM = 2MF$

Centroid



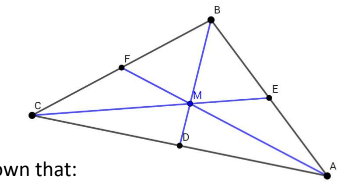
- Consider ΔABC and ΔEBF
- Since F is a midpoint, $BC = 2BF$
- Since E is a midpoint, $BA = 2BE$
- Also, $\angle ABC = \angle EBF$
- By the SAS rule, $\Delta ABC \sim \Delta EBF$
- Since $\Delta ABC \sim \Delta EBF$ and $BC = 2BF$, $AC = 2FE$

Centroid



- Consider ΔAMC and ΔFME
- By opposite angles, $\angle EMF = \angle AMC$
- By alternate angles, $\angle MAC = \angle MFE$
- By the AA rule, $\Delta AMC \sim \Delta FME$
- Since $AC = 2FE$ and $\Delta AMC \sim \Delta FME$, then $CM = 2ME$ and $AM = 2MF$

Centroid



- Thus, we have shown that:
  - Any two medians will **always** intersect at a point that splits the other median in a 1:2 ratio (Midpoint-Vertex)
- Hence, the third median BD must pass through M
  - Notice how point M splits all three medians in a 1:2 ratio (Midpoint-Vertex)
- Thus, the centroid exists and indeed it is point M

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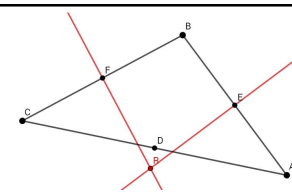
Circumcenter



- **Perpendicular bisector** of a line segment is the set of **all points** which are **equidistant** from the endpoints of that line segment
  - If point Y lies somewhere on the perpendicular bisector of BC, then $BY = CY$
  - If point Z is a point such that $BZ = CZ$, then Z **must** lie on the perpendicular bisector of BC

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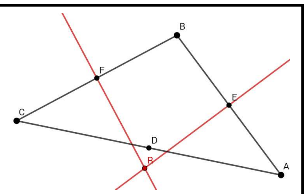
Circumcenter



- First, we arbitrarily draw two perpendicular bisectors starting from midpoints E and F
  - Let $\mathcal{L}_E$ be the perpendicular bisector passing through E
  - Let $\mathcal{L}_F$ be the perpendicular bisector passing through F
- Let R be the intersection of $\mathcal{L}_E$ and $\mathcal{L}_F$

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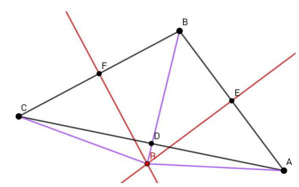
Circumcenter



- Since R lies on $\mathcal{L}_E$, $AR = BR$
- Since R lies on $\mathcal{L}_F$, $BR = CR$
- Since, $AR = BR$ and $BR = CR$, then $AR = CR$

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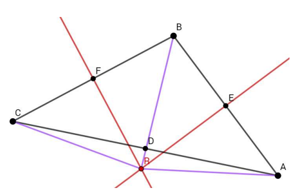
Circumcenter



- Since $AR = CR$, then R **must** lie on the perpendicular bisector of the line segment AC
- Hence, the third perpendicular bisector must always intersect at the same point that the other two perpendicular bisectors intersect.
- Thus, the circumcenter exists and indeed it is R

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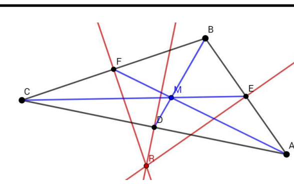
Orthocenter



- We will use the centroid and circumcenter that we proved previously to show that the orthocenter also lies on the same line.
  - Recall: The **centroid M** splits the **medians in a 2:1 ratio**
  - Recall: RF, RD, RE are the **perpendicular bisectors**

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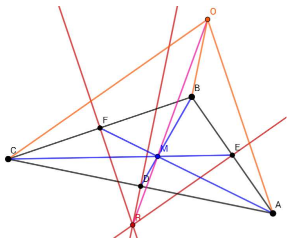
Orthocenter



- Extend line RM to a point O such that $MO = 2RM$
  - We want to use the property of the centroid later on
- Draw the line segments AO, BO, CO
- We will show that O is indeed the orthocenter

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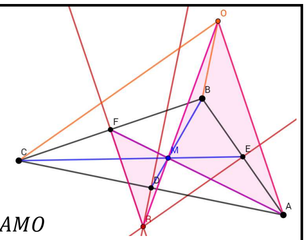
Orthocenter



- Consider ΔFMR and ΔAMO
- By construction, $OM = 2MR$
- From the centroid proof, $AM = 2MF$
- By opposite angles, $\angle FMR = \angle AMO$
- Hence, by SAS rule, $\Delta FMR \sim \Delta AMO$

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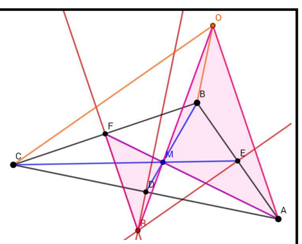
Orthocenter



- Since $\Delta FMR \sim \Delta AMO$, $\angle MFR = \angle MAO$
- Since $\angle MFR = \angle MAO$, $AO \parallel RF$ (alternate angles)
- Since $RF \perp BC$ and $AO \parallel RF$, then $AO \perp BC$
- Hence, AO is an altitude of the triangle

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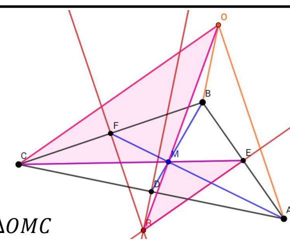
Orthocenter



- Consider ΔRME and ΔOMC
- By construction, $OM = 2MR$
- From the centroid proof, $CM = 2ME$
- By opposite angles, $\angle RME = \angle OMC$
- Hence, by SAS rule, $\Delta RME \sim \Delta OMC$

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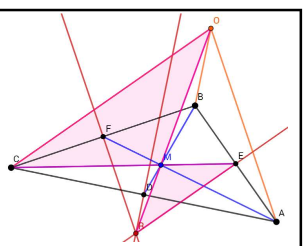
Orthocenter



- Since $\Delta RME \sim \Delta OMC$, $\angle MER = \angle MCO$
- Since $\angle MER = \angle MCO$, $CO \parallel RE$ (alternate angles)
- Since $RE \perp AB$ and $CO \parallel RE$, then $CO \perp AB$
- Hence, CO is an altitude of the triangle

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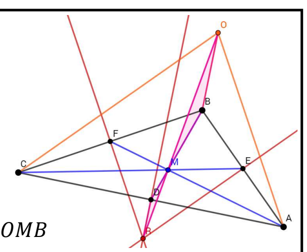
Orthocenter



- Consider ΔRMD and ΔOMB
- By construction, $OM = 2MR$
- From the centroid proof, $BM = 2MD$
- By opposite angles, $\angle RMD = \angle OMB$
- Hence, by SAS rule, $\Delta RMD \sim \Delta OMB$

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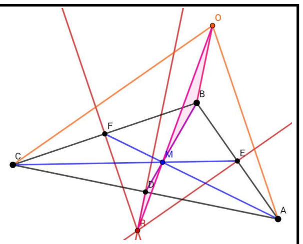
Orthocenter



- Since $\Delta RMD \sim \Delta OMB$, $\angle MRD = \angle MOB$
- Since $\angle MRD = \angle MOB$, $BO \parallel RD$ (alternate angles)
- Since $RD \perp AC$ and $BO \parallel RD$, then $BO \perp AC$
- Hence, BO is an altitude of the triangle

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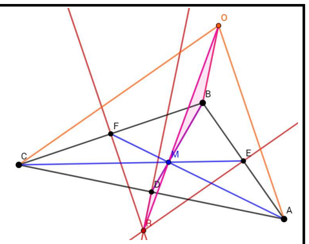
Orthocenter



- Since $\Delta RMD \sim \Delta OMB$, $\angle MRD = \angle MOB$
- Since $\angle MRD = \angle MOB$, $BO \parallel RD$ (alternate angles)
- Since $RD \perp AC$ and $BO \parallel RD$, then $BO \perp AC$
- Hence, BO is an altitude of the triangle

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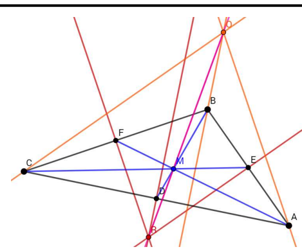
Orthocenter



- Since AO, BO, CO are all altitudes of the triangle and O is the point where all 3 altitudes intersect, then O must be the orthocenter
- Hence, the orthocenter exists and is indeed point O

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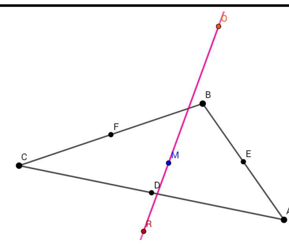
Euler Line



- But also, the orthocenter lies along the line RM which we constructed earlier
- Hence, the Euler Line exists and it is indeed RMO
  - In addition, we know from how we constructed this line that $MO = 2RM$

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Recap

- Proof Summary
  - Prove that the centroid exists
    - We did this using the ratio 2:1
  - Prove that the circumcenter exists
    - We did this using the definition of perpendicular bisectors
  - Prove that the orthocenter lies on the line formed by the centroid and circumcenter
    - We did this by extending the circumcenter-centroid line segment to twice its original length
- This completes the proof of the Euler Line

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So what?

- What are the applications of the Euler Line?
  - Well, you now know something most of your peers don't
- University-Level Mathematics
  - Applying mathematical concepts to real-life problems
  - Understanding and writing proofs about theorems
- Keys to Success
  - Try to generalize concepts
  - Try to explain and teach what you learned
  - Know when to study and when to take a break

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